Lecture on the second quantization

Sharareh Sayyad

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1 What we have learned from the first quantization: Fermionic system

A basis of the Fermionic N-particle Hilbert space are antisymmetric wave functions represented by the Slater determinant of the form

$$\text{Fermion: } \langle x|\Psi_{\alpha_1,\alpha_2\ldots\alpha_N}\rangle = \Psi_{\alpha_1,\alpha_2\ldots\alpha_N}(r_1,r_2,\ldots r_N) = \frac{1}{\sqrt{N!}}\det\begin{pmatrix}\phi_{\alpha_1}(r_1) & \ldots & \phi_{\alpha_1}(r_N)\\ \vdots & \ddots & \vdots \\ \phi_{\alpha N}(r_1) & \ldots & \phi_{\alpha_N}(r_N)\end{pmatrix}, \qquad (1)$$

where α is the index of the internal sub-Hilbert space (i.e. spin, orbital), and $\langle r|\alpha\rangle = \phi_\alpha(r)$.

This representation satisfies the important basis property which are orthonormality

$$\int d^3r_1\ldots d^3r_N \Psi_{\alpha_1,\alpha_2\ldots\alpha_N}(r_1,r_2,\ldots r_N)\Psi^*_{\alpha'_1,\alpha'_2\ldots\alpha'_N}(r_1,r_2,\ldots r_N) = \begin{cases}1 & \text{if } \{\alpha_1,\alpha_2\ldots\alpha_N\} = \{\alpha'_1,\alpha'_2\ldots\alpha'_N\}\\ 0 & \text{Otherwise}\end{cases} \qquad (2)$$

This representation is often known as the “first quantization”.

In short, this representation conveys two essential properties which are

1. **statistics of particles** (Fermionic statistics $\equiv$ antisymmetric wave function),
2. **occupation of a single particle state** (For Fermions: the Pauli exclusion principle imposes that either the state is occupied or not!).

This implies that by only knowing the statistics of particles, and an initial state (few-body or empty (vacuum) state) one can build the N-body Hilbert space.

2 Occupation number basis

Aside from the above representation of the basis of the N-particle Hilbert space, one can try to employ a more convenient wave function which encoded two important building block of the first quantization approach. In this approach instead of building the Hilbert space by figuring out “ each particle on which state (only occupied states will be considered to build the wave function)”, we would answer to “how many particles on each state (both occupied and unoccupied states are involved in constructing the wave function)”.

Here, we would claim that the N-particle wave function can be also introduced as

$$\text{Fermion: } |\Psi_{\alpha_1,\alpha_2\ldots\alpha_N}\rangle \overbrace{\equiv}^{\text{claim}} |n_1,n_2,\ldots,n_L\rangle \text{ where} \sum_{i=1}^{L} n_i = N, \qquad (3)$$

where L is the total number of states, and for the fermionic space $n_i \in \{0,1\}$ (due to the Pauli exclusion principle). We also would consider that such state has the norm of “one”. As the the total number of particles is fixed in the above state we would call that as the “occupation number basis”.

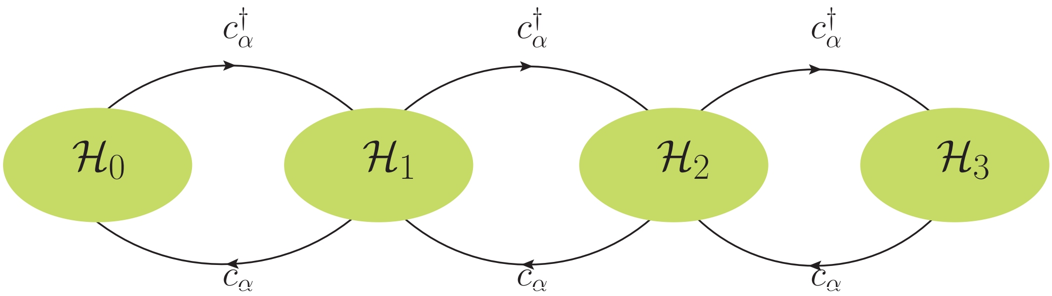

Figure 1: Schematic of Fock space. The n-body Hilbert space $\mathcal{H}_n$ is created by applying $c^\dagger$ to the basis of $n-1$-boy Hilbert space. The n-body Hilbert space $\mathcal{H}_n$ is annihilated by applying c to the its basis.

- **Vacuum state**

  The especial case where $N=0$ describes the vacuum state where

$$|\mathrm{Vac}\rangle = |0\rangle = |\Phi_{N=0}\rangle \equiv |0_1, 0_2, \dots 0_L\rangle \tag{4}$$

- **Single-particle state**

  Single-particle state is characterized by $N=1$ where

$$|\Phi_{N=1}\rangle \equiv |1_i, \dots, 0_j\rangle \text{where } i \in \{1 \dots L\}, \text{ and} j \neq i. \tag{5}$$

- **Creation and annihilation operators**

  This state can be constructed by using a creation operator such that

$$c_\alpha^\dagger|0\rangle = |\dots 0_{\alpha-1}, 1_\alpha, 0_{\alpha+1} \dots\rangle. \tag{6}$$

  This means that the creation operator ($c_\alpha^\dagger$) adds one particle at the single-particle state of α.

  Similarly by an annihilation operator , we can build the vacuum state as

$$c_\alpha|0\rangle \equiv |\dots 0_{\alpha-1}, 1_\alpha, 0_{\alpha+1} \dots\rangle = |0\rangle. \tag{7}$$

  This means that the annihilation operator (c_α) kills one particle at the single-particle state of α.

  One should note that applying the annihilation operator on the vacuum state would have the result of zero.

$$c_\alpha|0\rangle \equiv |\dots 0_{\alpha-1}, 0_\alpha, 0_{\alpha+1} \dots\rangle = 0. \tag{8}$$

- **Applying the creation and annihilation operators on an arbitrary N-particle state**

  The antisymmetric properties of the Fermionic wave function would be satisfied in the occupation number basis if for an arbitrary N-particle state of

$$|n_1, n_2, \dots n_\alpha, \dots\rangle \tag{9}$$

  we would enforce

$$c_\nu^\dagger|n_1, \dots n_\nu \dots\rangle = (-1)^{\sum_{\mu<\nu} n_\nu}(1-n_\nu)|n_1, \dots 1_\nu \dots\rangle, \tag{10}$$

$$c_\nu|n_1, \dots n_\nu \dots\rangle = (-1)^{\sum_{\mu<\nu} n_\mu} n_\nu|n_1, \dots 0_\nu \dots\rangle, \tag{11}$$

  where $\sum_{\mu<\nu} n_\mu$ counts the number of states(with indices less that ν) which were occupied. The $(1-n_\nu)$ ensures the satisfaction of the Pauli exclusion principle by not allowing the occupation of filled state ($n_\nu = 1$).

  By applying the creation (annihilation) operators the N-particle Hilbert space ($\mathcal{H}_N$) will be mapped to the $N+1(N-1)$-particle Hilbert space ($\mathcal{H}_{N+1}(\mathcal{H}_{N-1})$) which can be schematically drawn as

  c_ν and $c_\nu^\dagger$ are each other's adjoint in the Fock space

$$c_\nu^\dagger = (c_\nu)^\dagger, \tag{12}$$

$$c_\nu = (c_\nu^\dagger)^\dagger \tag{13}$$

  which maps states in the Fock space.

- **Construction of the N-body basis from the vacuum state**

  The collection of various M-particle Hilbert spaces, where $M = 0, 1 \dots N$, is known as the N-body Fock space.

  One can also construct the N-body state by acting the creation operators on the vacuum as

$$|n_1, n_2, \dots n_\alpha, \dots\rangle = (c_1^\dagger)^{n_1} \dots (c_\alpha^\dagger)^{n_\alpha} \dots |0\rangle. \tag{14}$$

- **Occupation number operator**

  We would define the number operator of state ν by $\hat{n}_\nu = c_\nu^\dagger c_\nu$ which its action on the N-body state is

$$\hat{n}_\nu = c_\nu^\dagger c_\nu |n_1, \dots, 0_\nu, \dots\rangle = 0, \tag{15}$$

$$\hat{n}_\nu = c_\nu^\dagger c_\nu |n_1, \dots, 1_\nu, \dots\rangle = (-1)^{\sum_{\mu<\nu} n_\nu} \underbrace{n_\nu}_{1} c_\nu^\dagger |n_1, \dots, 0_\nu, \dots\rangle, \tag{16}$$

$$= (-1)^{\sum_{\mu<\nu} n_\nu} n_\nu (-1)^{\sum_{\mu<\nu} n_\nu} (1 - \underbrace{0}_{c_\nu^\dagger |0_\nu\rangle}) |n_1, \dots, 1_\nu, \dots\rangle, \tag{17}$$

$$= n_\nu |n_1, \dots, n_\nu, \dots\rangle \tag{18}$$

  It is clear that the number operator $(\hat{N}_\nu)$ measures the occupation of the ν one-particle state

$$c_\nu c_\nu^\dagger |n_1, \dots, 1_\nu, \dots\rangle = 0, \tag{19}$$

$$c_\nu c_\nu^\dagger |n_1, \dots, 0_\nu, \dots\rangle = (-1)^{\sum_{\mu<\nu} n_\nu} (1 - \underbrace{n_\nu}_{0}) c_\nu |n_1, \dots, 1_\nu, \dots\rangle, \tag{20}$$

$$= (-1)^{\sum_{\mu<\nu} n_\nu} (1 - \underbrace{n_\nu}_{0}) (-1)^{\sum_{\mu<\nu} n_\nu} \underbrace{1}_{c_\nu |1_\nu\rangle} |n_1, \dots, 1_\nu, \dots\rangle, \tag{21}$$

$$= (1 - n_\nu) |n_1, \dots, n_\nu, \dots\rangle \tag{22}$$

  The total number operator is defined by

$$\hat{N} = \sum_\nu \hat{n}_\nu, \tag{23}$$

  which its action on the N-body state gives

$$\hat{N}|N\rangle = \hat{N}|n_1, \dots, n_\nu, \dots\rangle = \sum_\nu \hat{n}_\nu |n_1, \dots, n_\nu, \dots\rangle = \underbrace{\sum_\nu n_\nu}_{N} |n_1, \dots, n_\nu, \dots\rangle \tag{24}$$

  $|N\rangle$ is the eigenstate of total number operator,a nd any linear combination of N-particle state is also an eigenstate of $\hat{N}$ with eigenvalue N.

- **Commutation relations**

  Finally, the anticommutation relation of operators is reached as

$$\left(c_\nu^\dagger c_\nu + c_\nu c_\nu^\dagger\right) |n_1, \dots, n_\nu, \dots\rangle = (n_\nu + 1 - n_\nu) |n_1, \dots, n_\nu, \dots\rangle = |n_1, \dots, n_\nu, \dots\rangle \tag{25}$$

$$\{c_\nu, c_{\nu'}^\dagger\} = \delta_{\nu\nu'} \tag{26}$$

  Similarly, one can show that

$$\{c_\nu, c_{\nu'}\} = 0 \tag{27}$$

$$\{c_\nu^\dagger, c_{\nu'}^\dagger\} = 0 \tag{28}$$

$$c^\dagger_\nu c^\dagger_\mu |\ldots n_\mu \ldots n_\nu \ldots\rangle = \theta_\nu (1-n_\nu) c^\dagger_\nu |\ldots 1_\mu \ldots n_\nu \ldots\rangle \quad (29)$$
$$= \theta_\nu \theta_\mu (1-n_\nu)(1-n_\mu) |\ldots 1_\mu \ldots 1_\nu \ldots\rangle \quad (30)$$

$$c^\dagger_\mu c^\dagger_\nu |\ldots n_\mu \ldots n_\nu \ldots\rangle = \theta_\nu (-1)^{-n_\mu} (1-n_\nu) c^\dagger_\mu |\ldots n_\mu \ldots 1_\nu \ldots\rangle \quad (31)$$
$$= \theta_\nu \theta_\mu (1-n_\nu)(1-n_\mu)(-1)^{-1} |\ldots 1_\mu \ldots 1_\nu \ldots\rangle \quad (32)$$
$$= -\theta_\nu \theta_\mu (1-n_\nu)(1-n_\mu) |\ldots 1_\mu \ldots 1_\nu \ldots\rangle \quad (33)$$

where $\theta_\nu = (-1)^{\sum_{\beta<\nu} n_\beta}$ and $n_{\beta=\mu} = 1$. The factor $(-1)^{-n_\mu}$ has been added to extract the extra minus sign (at the time where this factor was evaluated $n_\mu = 0$ in Eq. (33)).

- **Identity operator**

  One should note that superpositions of the introduced basis can construct the first quantization wave function and consequently the identity operator of a N-particle basis yields

$$\mathbb{I}_N = \sum_{n_1=0}^{1} \sum_{n_2=0}^{1} \cdots \sum_{n_L=0}^{1} |n_1, n_2, \ldots, n_L\rangle\langle n_1, n_2, \ldots, n_L| \text{ where} \sum_{i=1}^{L} n_i = N. \quad (34)$$

- **Basis transformation**

  Assume that we have two complete, orthonormal single-particle basis functions of $\{\phi_l(r)\}$ $(\{|\phi_l\rangle\})$, and $\{\rho_\alpha(r)\}$ $(|\rho_\alpha\rangle)$. One can expand one basis in terms of the other as

$$\rho_\alpha(r) = \sum_\alpha \phi_l(r) U_{l\alpha}, \quad (35)$$

where

$$U_{l\alpha} = \langle \phi_l | \rho_\alpha \rangle. \quad (36)$$

One should note that the applied transformation U is unitary,

$$(UU^\dagger)_{ml} = \sum_\alpha \langle \phi_m | \rho_\alpha \rangle \langle \rho_\alpha | \phi_l \rangle, \quad (37)$$
$$= \langle \phi_m | \phi_l \rangle, \quad \text{where} \sum_\alpha |\rho_\alpha\rangle\langle \rho_\alpha| = \mathbb{I}, \quad (38)$$
$$= \delta_{ml}. \quad (39)$$

Now let $c^\dagger_l$ creates a particle in the orbital $|\phi_l\rangle$, and $d^\dagger_\alpha$ creates a particle in orbital $|\rho_\alpha\rangle$. This means that Eq. (35) which has the form of

$$|\rho_\alpha\rangle = \sum_\alpha |\phi_l\rangle U_{l\alpha}, \quad (40)$$

can be written as

$$d^\dagger_\alpha |0\rangle = \sum_\alpha c^\dagger_l |0\rangle U_{l\alpha}, \quad (41)$$

resulting in

$$d^\dagger_\alpha = \sum_\alpha c^\dagger_l U_{l\alpha}, \quad (42)$$
$$d_\alpha = \sum_\alpha U^*_{l\alpha} c_l, \quad \text{By taking the Hermitian conjugate of the upper equation.} \quad (43)$$
$$d_\alpha = \sum_\alpha (U^\dagger)_{\alpha l} c_l. \quad (44)$$

This basis transformation would not change the particle statistics. This can be easily can be checked by

$$\{d_\alpha, d^\dagger_\beta\} = \sum_{lm} U^*_{l\alpha} U_{m\beta} \underbrace{\{c_l, c^\dagger_m\}}_{\delta_{ml}} = \sum_m U^*_{m\alpha} U_{m\beta} = \delta_{\alpha,\beta}. \quad (45)$$

- **Action of one (two)-body operators in Fock space**

  For the single particle basis function of $\{|\phi_\alpha\rangle\}$ which is created by the operator $|\phi_\alpha\rangle = c^\dagger_\alpha|0\rangle$, the one and two body operators can be written as

$$\text{One-body operator: } \hat{\mathcal{O}}^{(1)} = \sum_{\alpha,\beta} |\phi_\alpha\rangle\langle\phi_\alpha|\hat{\mathcal{O}}^{(1)}|\phi_\beta\rangle\langle\phi_\beta| \Leftrightarrow \quad \hat{\mathcal{O}}^{(1)} = \sum_{\alpha,\beta}\langle\alpha|\hat{\mathcal{O}}^{(1)}|\beta\rangle c^\dagger_\alpha c_\beta, \tag{46}$$

(47)

  where with one-body orbitals of $\langle r|\beta\rangle = \phi_\beta(r)$ the above matrix element would be

$$\langle\alpha|\hat{\mathcal{O}}^{(1)}|\beta\rangle = \int \mathrm{d^3r}\phi^*_\alpha(r)\hat{h}(r)\phi_\beta(r) \tag{48}$$

$$\text{Two-body operator: } \hat{\mathcal{V}}^{(2)} = \sum_{\alpha,\beta,\gamma,\delta} |\phi_\alpha,\phi_\beta\rangle\langle\phi_\alpha,\phi_\beta|\hat{\mathcal{V}}|\phi_\gamma,\phi_\delta\rangle\langle\phi_\gamma,\phi_\delta| \Leftrightarrow \hat{\mathcal{V}} = \frac{1}{2}\sum_{\alpha,\beta,\gamma,\delta}\langle\alpha,\beta|\hat{\mathcal{V}}|\gamma,\delta\rangle c^\dagger_\alpha c^\dagger_\beta c_\delta c_\gamma, \tag{49}$$

  where the factor of 1/2 is added to avoid the double counting as swapping both creation and annihilation sets of operators is giving the same result, and the two-body matrix element has the explicit form of

$$\langle\alpha,\beta|\hat{\mathcal{V}}|\gamma,\delta\rangle = \iint \mathrm{drdr'}\phi^*_\alpha(r)\phi^*_\beta(r')\mathcal{V}(r-r')\phi_\gamma(r)\phi_\delta(r'). \tag{50}$$

3 Examples and applications

1. Mapping the first quantization to second quantization basis

   A Fock space with the two particles is a combination of zero-particle, one-particle, ad two-particle Hilbert spaces. If we assume that the single-particle wave functions for particles "1" and "2" in the first quantization are ϕ_1, ϕ_2, then the second quantization representation of the occupation basis would be

total occupation number	first quantization		2nd quantization ($|n_1,n_2\rangle$)
0	$|0\rangle$	$\Leftrightarrow$	$|0\rangle$
1	ϕ_1	$\Leftrightarrow$	$|n_1,0\rangle$
1	ϕ_2	$\Leftrightarrow$	$|0,n_2\rangle$
2	$\underbrace{\phi_{12}}_{\frac{1}{\sqrt{2}}[\phi_1(r_1)\phi_2(r_2)-\phi_2(r_1)\phi_1(r_2)]}$	$\Leftrightarrow$	$|n_1,n_2\rangle$

2. Mapping the one-body operator into the momentum basis

   We have shown that the general form of the one-body operator has the form of

$$\mathcal{O} = \sum_{R,R'} |\phi_R\rangle\langle\phi_R|\hat{\mathcal{O}}^{(1)}|\phi_{R'}\rangle\langle\phi_{R'}| \tag{51}$$

$$= \sum_{R,R'} c^\dagger_R\langle R|\hat{\mathcal{O}}^{(1)}|R'\rangle c_{R'}, \tag{52}$$

   where $|\phi_R\rangle$ is single-particle basis function and $|R\rangle$ is the single-particle occupation number representation of that in the Fock space.

   Employing the transformation of the form

$$c^\dagger_k = \frac{1}{\sqrt{N}}\sum_R e^{ik.R}c^\dagger_R \quad c^\dagger_R = \frac{1}{\sqrt{N}}\sum_{k\in 1BZ} e^{-ik.R}c^\dagger_k, \tag{53}$$

$$c_k = \frac{1}{\sqrt{N}}\sum_R e^{-ik.R}c_R \quad c_R = \frac{1}{\sqrt{N}}\sum_{k\in 1BZ} e^{ik.R}c_k \tag{54}$$

The second quantization representation of the one-body operator would be

$$\mathcal{O} = \sum_{R,R'} c_R^\dagger \langle R|\hat{\mathcal{O}}^{(1)}|R'\rangle c_{R'}, \tag{55}$$

$$= \sum_{k,k'} c_k^\dagger \underbrace{\sum_R \frac{1}{\sqrt{N}} e^{-ik.R} \langle R|}_{\langle k|} \hat{\mathcal{O}}^{(1)} \underbrace{\sum_{R'} |R'\rangle \frac{1}{\sqrt{N}} e^{ik'.R'}}_{|k'\rangle} c_{k'}, \tag{56}$$

$$= \sum_{k,k'} c_k^\dagger \langle k|\hat{\mathcal{O}}^{(1)}|k'\rangle c_{k'} \tag{57}$$

– $\hat{\mathcal{O}}^{(1)} = \frac{p^2}{2m}$ then in three dimension we would have

$$\langle \overrightarrow{k}|\hat{\mathcal{O}}^{(1)}|\overrightarrow{k'}\rangle = \frac{\hbar(k_x^2 + k_y^2 + k_z^2)}{2m} \delta_{\overrightarrow{k},\overrightarrow{k'}}, \tag{58}$$

which results in

$$\mathcal{O} = \sum_{\overrightarrow{k},\overrightarrow{k'}} c_{\overrightarrow{k}}^\dagger \langle \overrightarrow{k}|\hat{\mathcal{O}}^{(1)}|\overrightarrow{k'}\rangle c_{\overrightarrow{k'}}, \tag{59}$$

$$= \sum_{\overrightarrow{k},\overrightarrow{k'}} c_{\overrightarrow{k}}^\dagger \frac{\hbar(k_x^2 + k_y^2 + k_z^2)}{2m} \delta_{\overrightarrow{k},\overrightarrow{k'}} c_{\overrightarrow{k'}}, \tag{60}$$

$$= \sum_{\overrightarrow{k}} \frac{\hbar(k_x^2 + k_y^2 + k_z^2)}{2m} c_{\overrightarrow{k}}^\dagger c_{\overrightarrow{k}}, \tag{61}$$

$$= \sum_{\overrightarrow{k}} \frac{\hbar(k_x^2 + k_y^2 + k_z^2)}{2m} n_{\overrightarrow{k}} \quad \text{kinetic energy of fermions with mass } m \tag{62}$$

3. Tight binding Hamiltonian

For a system with the single particle Wannier orbital of $|\phi_R\rangle$, where Ron the lattice with N sites, the tight binding Hamiltonian can be written as

$$H_{TB} = \sum_{R,R'} |\phi_R\rangle \underbrace{\int d^3r \phi_R^*(r)[-\frac{\hbar^2}{2m}\nabla^2 + V(r)]\phi_{R'}(r)}_{h(R-R')} \langle \phi_{R'}| \tag{63}$$

$$H_{TB} = \sum_{R,R'} c_R^\dagger h(R - R') c_{R'} \tag{64}$$

One-body space-nonlocal operator

Diagonalization by Fourier transform of operators

$$c_k^\dagger = \frac{1}{\sqrt{N}} \sum_R e^{ik.R} c_R^\dagger \quad c_R^\dagger = \frac{1}{\sqrt{N}} \sum_{k\in 1BZ} e^{-ik.R} c_k^\dagger, \tag{65}$$

$$c_k = \frac{1}{\sqrt{N}} \sum_R e^{-ik.R} c_R \quad c_R = \frac{1}{\sqrt{N}} \sum_{k\in 1BZ} e^{ik.R} c_k \tag{66}$$

$c_k, c_k^\dagger$ also fermionic operators which satisfy anticommutation relation

$$\{c_k, c_{k'}^\dagger\} = \delta_{k,k'}. \tag{67}$$

Holds for any unitary transformation.

$$H = \sum_{RR'} c_R^\dagger h_{R-R'} c_{R'}, \tag{68}$$
$$= \frac{1}{N} \sum_{k,k'} \sum_{R,R'} e^{-ik.R} e^{ik'.R'} c_k^\dagger c_{k'} h_{R-R'}, \tag{69}$$
$$= \frac{1}{N} \sum_{k,k'} \underbrace{\sum_{R'} e^{-i(k'-k).R'}}_{N\delta_{k,k'}} \underbrace{(\sum_{R-R'} e^{-ik.(R-R')} h_{R-R'})}_{\epsilon_k} c_k^\dagger c_{k'}, \tag{70}$$
$$= \sum_k \epsilon_k c_k^\dagger c_k \tag{71}$$

Eigenstates are occupation numbers states where $\prod_k (c_k^\dagger)^{n_k}|0\rangle$, where the ground state included all states below the Fermi wave-number which are occupied, $|\Phi\rangle = \prod_{k:\epsilon_k<E_F} c_k^\dagger|0\rangle$.

4 Second quantization of bosons

The occupation number representation of bosons is defined as

$$|n_1, n_2, n_3, \ldots\rangle \equiv |N\rangle \quad \text{where} \sum_i n_i = N, \tag{72}$$

and despite fermions, n_i does not have any restrictions in value but to be positive. Similar like fermions, there is one-to-one correspondence between the above second-quantized wave-function and the N-body wave-function in the first quantization representation

$$\Psi_{sym}(r_1, r_2 \ldots r_N) = \frac{1}{\sqrt{N! \prod_\nu n_\nu!}} \sum_{P\in S_N} P\phi_{\nu_1}(r_1)\phi_{\nu_2}(r_2)\ldots\phi_{\nu_N}(r_N), \tag{73}$$

where we sum over all possible permutation (P) in the set of symmetric permutations (S_N).

- **Vacuum state**

  Similar as Fermions, the zero-body basis is known as the vacuum state which is

$$|\text{Vac}\rangle = |0\rangle = |0_1, 0_2, \ldots, 0_N\rangle. \tag{74}$$

- **Creation and annihilation operators**

  The bosonic creation and annihilation operators (analog of the creation and annihilation operators of harmonic oscillator) are defined as

$$b_\nu^\dagger|\ldots, n_\nu, \ldots\rangle = \sqrt{n_\nu + 1}|\ldots, n_\nu + 1, \ldots\rangle \tag{75}$$
$$b_\nu|\ldots, n_\nu, \ldots\rangle = \sqrt{n_\nu}|\ldots, n_\nu - 1, \ldots\rangle \tag{76}$$

- **Number operator**

  The number operator of the single-particle state is defined as

$$\underbrace{b_\nu^\dagger b_\nu}_{n_\nu}|\ldots, n_\nu, \ldots\rangle = \sqrt{n_\nu} b_\nu^\dagger|\ldots, n_\nu - 1, \ldots\rangle = (\sqrt{n_\nu})^2|\ldots, n_\nu, \ldots\rangle. \tag{77}$$

  The total number operator is defined by

$$\hat{N} = \sum_\nu n_\nu, \tag{78}$$

  which its action on the N-body occupation number state has the result of

$$\hat{N}|n_1 \ldots, n_\nu, \ldots n_N\rangle = \sum_\nu n_\nu|n_1 \ldots, n_\nu, \ldots n_N\rangle = N|n_1 \ldots, n_\nu, \ldots n_N\rangle. \tag{79}$$

- **Construction of the N-body basis from the vacuum state**

  The normalized N-body basis can be written as

$$|n_1 \ldots, n_\nu, \ldots n_N\rangle = \prod_\nu \frac{(b_\nu^\dagger)^{n_\nu}}{\sqrt{n_\nu!}}|0\rangle \tag{80}$$

- **Commutation relations for bosons**

  The bosonic commutation relations are

$$[b_\mu, b_\nu] = 0, \tag{81}$$
$$[b_\mu^\dagger, b_\nu^\dagger] = 0, \tag{82}$$
$$[b_\mu, b_\nu^\dagger] = \delta_{\mu,\nu}, \tag{83}$$

  which has been obtained by employing the orthonormality of basis, as well as Eqs. (75, 76).

- **Representation on one- and two-body operators**

  As the representation independent of statistics of involved particles, our derived representation of the operators for Fermionic systems is also valid for bosons.

$$\hat{\mathcal{O}}^{(1)} = \sum_{\alpha,\beta}\langle\alpha|\hat{\mathcal{O}}|\beta\rangle b_\alpha^\dagger b_\beta, \tag{84}$$
$$\hat{\mathcal{V}}^{(2)} = \frac{1}{2}\sum_{\alpha,\beta,\gamma,\delta}\langle\alpha,\beta|\hat{\mathcal{V}}|\gamma\delta\rangle b_\alpha^\dagger b_\beta^\dagger b_\delta b_\gamma. \tag{85}$$